# Spectral Theory and its Applications -
# - Spectral and Functional Inequalities

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# Lecture 1

# Plan

- Functional inequalities (Sobolev inequalities, Hardy inequalities)
- Spectrum of Dirichlet Laplacian
- Some basic facts from Spectral Theory of Schrödinger operators
- Lieb-Thirring inequalities
- Calogero inequalities
- Spectrum of functional-difference operators

It is assumed that the students passed the standard Functional Analysis course + a course in Theory of Distribution.

- Recommended literature:

Book: "Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities" by R.Frank, A.Laptev and T.Weidl, see

• Hardy inequality on $\mathbb{R}_+$.

Let $\mathbb{R}_+ = (0, \infty)$.

**Theorem.** For any $u \in H_0^1(\mathbb{R}_+)$ the following bound holds true

$$\frac{1}{4} \int_0^\infty \frac{|u(x)|^2}{x^2}\, dx \le \int_0^\infty |u'(x)|^2\, dx.$$

*Proof.* We begin by considering functions in $C_0^\infty(\mathbb{R}_+)$. Integration by part we obtain

$$\begin{aligned}\int_0^\infty \frac{|u(x)|^2}{x^2}\, dx &\le -\frac{|u(x)|^2}{x}\Big|_0^\infty + 2\mathrm{Re} \int_0^\infty \overline{u(x)}\, u'(x)\, x^{-1}\, dx \\ &\le 2 \Big( \int_0^\infty \frac{|u(x)|^2}{x^2}\, dx \Big)^{1/2} \Big( \int_0^\infty |u'(x)|^2\, dx \Big)^{1/2}.\end{aligned}$$

The constant on the right side is optimal but never achieved.

- Hardy inequality on $\mathbb{R}^d$, $d \geq 3$.

**Theorem.** Let $d \geq 3$. Then for any $u \in H^1(\mathbb{R}^d)$ we have

$$\left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx.$$

*Proof.* Let $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} 0 \leq Q^*Q &= \left(\nabla - \alpha \frac{x}{|x|^2}\right)\left(-\nabla - \alpha \frac{x}{|x|^2}\right) \\ &= -\Delta - \alpha \nabla \frac{x}{|x|^2} + \alpha \frac{x}{|x|^2}\nabla + \frac{\alpha^2}{|x|^2} = -\Delta - \alpha \frac{d-2}{|x|^2} + \frac{\alpha^2}{|x|^2}. \end{aligned}$$

Letting $\alpha = (d-2)/2$ we find

$$-\Delta \geq \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2}.$$

The latter should be understood in the sense of quadratic forms, namely

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx = (-\Delta u, u)_{L^2(\mathbb{R}^d)} \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx.$$

The proof is complete.

• Non-increasing rearrangement.

*Definition.* The distribution function of a Lebesgue measurable function $u$ on an open subset $\Omega \subset \mathbb{R}^d$ is the map $\mu_u : [0,\infty) \to [0,\infty)$ defined by

$$\mu_u(\lambda) = \text{meas}\,\{x \in \Omega : \, |u(x)| > \lambda\}.$$

The non-increasing rearrangement of $u$ is the function $u^* : [0,\infty) \to [0,\infty)$ defined by

$$u^*(t) = \inf\{\lambda \in [0,\infty) : \, \mu_u(\lambda) \le t\}.$$

The rearrangement preserves the $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$ class of functions and

$$\|u\|_{L^p(\mathbb{R}^d)} = \|u^*\|_{L^p(\mathbb{R}^d)}.$$

**Theorem.** (Pólya-Szegö) Let $1 \le p$. Then

$$\|\nabla u^*\|_{L^p(\mathbb{R}^d)} \le \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

• Sobolev inequality.

Hardy inequality implied Sobolev inequality.

**Theorem.** Let $d \geq 3$ and $q = 2d/(d-2)$. Then there is a constant $S_d$ such that

$$\left( \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \right)^{q/2} \geq S_d \int_{\mathbb{R}^d} |u|^q \, dx.$$

• Sobolev inequality.

Hardy inequality implied Sobolev inequality (R. Seiringer '09)

**Theorem.** Let $d \geq 3$ and $q = 2d/(d-2)$. Then there is a constant $S_d$ such that

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2 \, dx\right)^{q/2} \geq S_d \int_{\mathbb{R}^d} |u|^q \, dx.$$

*Proof.* Note that $|\nabla |u|| \leq |\nabla u|$. Therefore it is enough to consider $u \geq 0$. Since $\|u\|_{L^p(\mathbb{R}^d)} = \|u^*\|_{L^p(\mathbb{R}^d)}$ and $\|\nabla u^*\|_{L^p(\mathbb{R}^d)} \leq \|\nabla u\|_{L^p(\mathbb{R}^d)}$ we can consider $u$ that are non-negative, radial and non-increasing.

For any $y \in \mathbb{R}^d$ and $0 < a < 1$ we have

$$\int_{\mathbb{R}^d} u^q(x) \, dx \geq \int_{a|y| \leq |x| \leq |y|} u^q \, dx \geq C_d |y|^d u^q(y)$$

with $C_d = (1-a)^n / |\mathbb{S}^{d-1}|$. Thus

$$\begin{aligned}\left(\int_{\mathbb{R}^d} u^q(x) \, dx\right)^{2/d} u^2(y) \, |y|^{-2} &\geq C_d^{2/d} |y|^2 u^{2q/d}(y) \, u^2(y) \, |y|^{-2} \\ &= C_d^{2/d} \, u^{2q/d+2}(y).\end{aligned}$$

Choosing $q = 2^* = 2d/(d-2)$ we obtain $2q/d + 2 = 2^*$ and we find

$$C_d^{2/d} \int_{\mathbb{R}^d} u^{2^*}(y) \, dy \leq \left(\int_{\mathbb{R}^d} u^{2^*}(y) \, dy\right)^{2/d} \int_{\mathbb{R}^d} \frac{u^2(y)}{|y|^2} \, dx.$$

Finally by using Hardy's inequality we conclude

$$C_d^{2/d} \left(\int_{\mathbb{R}^d} u^{2^*}(y) \, dy\right)^{1-2/d} \leq \left(\frac{2}{d-2}\right)^2 \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx.$$

• Sobolev inequality does not imply Hardy inequality, see
book "The Geometry and Analysis of Hardy's Inequality" by A .A. Balinsky, W.D. Evans, R.T. Lewis.

Indeed, the reverse implication can be obtained only in terms of the weak spaces $L^{q,\infty}(\mathbb{R}^d)$ defined as the space of Lebesgue measurable functions $u$ on $\mathbb{R}^d$ such that

$$\|u\|_{L^{2,\infty}(\mathbb{R}^d)} = \sup_{t>0} t^2\, \mu_u(t) < \infty, \qquad t > 0,$$

where $\mu_u(t) = \text{meas}\,\{x \in \mathbb{R}^d :\, |u(x)| > t\}$.

It is easy to show that

$$\|u\|_{L^{2,\infty}(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{L^{2,\infty}(\mathbb{R}^d)} = \|u^*\|_{L^{2,\infty}(\mathbb{R}^d)}.$$

From Sobolev's inequality we can only obtain

$$\left\|\frac{u}{|\cdot|}\right\|_{L^{2,\infty}(\mathbb{R}^d)} \leq \left(\frac{d-2}{2}\right)^{1/d} \left(\frac{d}{d-2}\right)^{1/2} \omega_{d-1} \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx, \quad d > 2.$$

- Corollary of Sobolev's inequality.

**Corollary.** There is a constant $C_d > 0$ such that for any open set $\Omega \subset \mathbb{R}^d$ of finite measure and any $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{C_d}{|\Omega|^{2/d}} \int_\Omega |u|^2 \, dx.$$

*Proof.* Using Hölder's inequality we obtain

$$\int_\Omega |u|^2 \, dx \leq |\Omega|^{d/2} \left( \int_{\mathbb{R}^d} |u|^{2d/(d-2)} \, dx. \right)^{(d-2)/d} \leq S_d \, |\Omega|^{d/2} \int_\Omega |\nabla u|^2 \, dx.$$

• Dirichlet Laplacian.

Consider the Dirichlet Laplacian $-\Delta_\Omega^D$ in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a domain of finite measure

$$Hu = -\Delta_\Omega^D u = \lambda u, \qquad u|_{\partial\Omega} = 0.$$

The operator $H$ is defined by its quadratic form

$$(Hu, u)_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx,$$

defined on $H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}$.

• Faber–Krahn inequality.

**Proposition.** Let $\lambda_1$ be the lowest eigenvalue of the Dirichlet Laplacian. Then

$$\lambda_1(-\Delta_\Omega^D) \geq \frac{C_d}{|\Omega|^{2/d}}.$$

*Remark.* The optimal constant occurs when $\Omega$ is a ball and can be expressed in terms of a zero of a Bessel function.

In particular we also obtain

**Corollary.** Let $\lambda_1$ be the lowest eigenvalue of the Dirichlet Laplacian $-\Delta_\Omega^D$. Then

$$\lambda_1(-\Delta_\Omega^D) > 0.$$

• Magnetic Hardy inequality with AB-vector potetial in $\mathbb{R}^2$.

Let introduce an Aharonov-Bohm vector potential

$$A(x) = \alpha \left( \frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

**Theorem.** Let $u \in H_0^1(\mathbb{R}^2 \setminus \{0\})$. Then

$$\int_{\mathbb{R}^2} |(i\nabla - A(x))u(x)|^2 \, dx \geq \min_{n \in \mathbb{Z}} |n - \alpha|^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} \, dx.$$

*Proof.* Let $(r,\theta)$ be polar coordinates in $\mathbb{R}^2$. Consider the Fourier decomposition

$$u(r,\theta)=\sum_{n\in\mathbb{Z}}u_n(r)e^{-in\theta}.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^2}|(i\nabla-A(x))u(x)|^2\,dx &= \int_0^\infty\int_{-\pi}^{\pi}\left(|u_r'|^2+\frac{|(i\partial_\theta-\alpha)u|^2}{r^2}\right)r\,d\theta dr\\
&\geq 2\pi\int_0^\infty\sum_{n\in\mathbb{Z}}\frac{|n-\alpha|^2|u_n(r)|^2}{r}\,dr\\
&\geq \min_{n\in\mathbb{Z}}|n-\alpha|^2\,2\pi\int_0^\infty\sum_{n\in\mathbb{Z}}\frac{|u_n(r)|^2}{r}\,dr=\min_{n\in\mathbb{Z}}|n-\alpha|^2\int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x|^2}\,dx.
\end{aligned}$$

The proof is complete.

• Hardy's inequality on antisymmetric functions.

By $H^1_A(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ we denote a class of antisymmetric functions satisfying the following antisymmetry conditions:

$$u(\dots, x_i, \dots, x_j, \dots) = -u(\dots, x_j, \dots, x_i, \dots),$$

where $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^d$.

Consider polar coordinates $x = (r, \theta)$, $r \in (0, \infty)$, and $\theta \in \mathbb{S}^{d-1}$. Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right) r^{d-1} \, d\theta dr.$$

Let $\mathcal{Y}$ be the orthonormal system of spherical harmonic functions on $\mathbb{S}^{d-1}$ and let $\mathcal{Y}_A \subset \mathcal{Y}$ be the orthonormal subset of the set $\mathcal{Y}$ that are restrictions of totally antisymmetric homogeneous harmonic polynomials. For any $u \in H^1_A(\mathbb{R}^N)$ we have

$$u(r, \theta) = \sum_{k: Y_k \in \mathcal{Y}_A} u_k(r) Y_k(\theta).$$

Denote by $M(d)$ the smallest integer so that there is an antisymmetric harmonic homogeneous polynomial $P_{M(d)} \not\equiv 0$ with degree $M(d)$.

One of such harmonic polynomials could be represented by the Vandermonde determinant

$$\mathcal{V}_d(x) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_d \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_d^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{d-1} & x_2^{d-1} & x_3^{d-1} & \dots & x_d^{d-1} \end{vmatrix}.$$

that implies that

$$M(d) = \frac{d(d-1)}{2}$$

Denote by $Y_{M(d)}$ the restriction of $\mathcal{V}_d$ to $\mathbb{S}^{d-1}$. Then if $-\Delta_\theta$ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ we have

$$-\Delta_\theta Y_{M(d)}(x) = M(d)(M(d)+d-2)\, Y_{M(d)}(x) = \lambda_{M(d)}\, Y_{M(d)}(x),$$

where

$$\lambda_{M(d)} = \frac{d(d-1)(d^2+d-4)}{4}.$$

**Theorem.** (Th. Hoffmann-Ostenhof & AL, '21)

Let $u \in H^1_A(\mathbb{R}^d)$, $d \geq 2$. Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq C_A(d) \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx,$$

where

$$C_A(d) = \lambda_{M(d)} + \frac{(d-2)^2}{4} = \frac{(d^2-2)^2}{4}.$$

*Remark.* If $d = 2$ then the lowest eigenvalue on the Laplace-Beltrami operator on the circle equals one and therefore for functions $u(x_1, x_2) = -u(x_2, x_1)$ we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx.$$

*Remark.* Note that the constant $C_A(d) \sim d^4/4$ compared with the classical Hardy's constant that grows as $d^2/4$, as $d \to \infty$.

*Proof.* Using polar coordinates we have

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right) r^{d-1} \, d\theta dr.$$

For any $u \in H^1_A(\mathbb{R}^d)$ we consider $u(r,\theta) = \sum_{k: Y_k \in \mathcal{Y}_A} u_k(r) Y_k(\theta)$. Then

$$\int_{\mathbb{S}^{d-1}} |\nabla_\theta u(r,\theta)|^2 \, d\theta = \sum_{k=M(d)}^{\infty} \lambda_k |u_k(r)|^2$$
$$\geq \lambda_{M(d)} \sum_{k=M(d)}^{\infty} |u_k(r)|^2 = \lambda_{M(d)} \int_{\mathbb{S}^{d-1}} |u(r,\theta)|^2 \, d\theta.$$

For the radial part we use the classical Hardy inequality on the half-line

$$\int_0^\infty \left| \frac{\partial u}{\partial r} \right|^2 r^{d-1} \, dr \geq \frac{(d-2)^2}{4} \int_0^\infty \frac{|u|^2}{r^2} r^{d-1} \, dr.$$

Finally we arrive at

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq \int_0^\infty \int_{\mathbb{S}^{d-1}} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \lambda_{M(d)} \frac{|u(r,\theta)|^2}{r^2} \right) r^{d-1} \, d\theta dr$$
$$\geq \left( \lambda_{M(d)} + \frac{(d-2)^2}{4} \right) \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx.$$

*Open Problem.* What is the best constant in the Sobolev inequality on antisymmetric functions.

- Versions of Hardy-Sobolev type inequalities.

**Theorem.** Let $u \in H^1_A(\mathbb{R}^2)$ and let $0 \leq \vartheta < 1$. Then there is a constant $C_{2,\vartheta}$ such that

$$C_{2,\vartheta} \left( \int_{\mathbb{R}^2} \frac{|u|^{\frac{2}{1-\vartheta}}}{|x|^2} dx \right)^{1-\vartheta} \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx.$$

**Theorem.** Let $d \geq 3$, $p = \frac{2d}{d-2\vartheta}$, $0 \leq \vartheta \leq 1$ and $\gamma = 2d\, \frac{\vartheta-1}{d-2\vartheta}$. Then

$$C_{d,\vartheta} \left( \int_{\mathbb{R}^d} |x|^\gamma |u|^p dx \right)^{2/p} \leq \int_{\mathbb{R}^d} |\nabla u|^2 dx, \quad u \in H^1_A(\mathbb{R}^d),$$

with some $C_(d,\vartheta) > 0$.

Thank you