Spectral Theory and its Applications -
- Spectral and Functional Inequalities

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Lecture 2

• Pólya's conjecture.

Let $\Omega \subset \mathbb{R}^d$ be a domain of finite measure and let $-\Delta_\Omega^D$ be the Dirichlet Laplacian in $L^2(\Omega)$ defined via it quadratic form

$$\int_\Omega |\nabla u|^2 \, dx, \qquad u \in H_0^1(\Omega).$$

The famous conjecture of Pólya ('54) states that the number of eigenvalues λ_k of $-\Delta_\Omega^D$ below λ of the Dirichlet Laplacian satisfy

$$\begin{aligned} N(\lambda, -\Delta_\Omega^D) = \#\{k : \lambda_k < \lambda\} &\le (2\pi)^{-d} \int_\Omega \int_{|\xi|^2<\lambda} d\xi dx \\ &= (2\pi)^{-d} \, |\Omega| \, \lambda^{d/2} \int_{|\xi|^2<1} d\xi = L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}. \end{aligned}$$

where $L_{0,d}^{cl} := (2\pi)^{-d}|\mathbb{B}_d| = (2\pi)^{-d} \int_{|\xi|<1} d\xi$.
Equivalently

$$\lambda_k \ge \left(\frac{(2\pi)^d}{|\Omega||\mathbb{B}_d|} \right)^{2/d} k^{2/d}, \qquad k = 1, 2, \dots .$$

- Weyl's asymptotics.

The conjecture was motivated by the Weyl's asymptotic formula that stated

$$N(\lambda, -\Delta_\Omega^D) = L_{0,d}^{cl}\, \lambda^{d/2}|\Omega| + o(\lambda^{d/2}), \quad \text{as} \quad \lambda \to \infty,$$

"phase volume" type asymptotics.

In order to prove it Weyl used a version of the max-min principle, namely Dirichlet-Neumann bracketing for cubes

$$N(\lambda, -\Delta_Q^D) \le N(\lambda, -\Delta_Q) \le N(\lambda, -\Delta_Q^N).$$

If $\Omega \subset R^2$ for each square with side a we find that the eigenvalues are equal to
$\{\lambda_{nm}^{\mathbb{D}}(a) = \pi^2 a^{-2}(n^2 + m^2) : \, n, m = 1, 2, 3, \dots\}$ for the Dirichlet problem
and
$\{\mu_{nm}^{\mathbb{N}}(a) = \pi^2 a^{-2}(n^2 + m^2) : \, n, m = 0, 1, 2, 3, \dots\}$ for the Neumann problem.

Counting

$$\#\{(n,m) : \, \lambda_{nm}^{\mathbb{D}}(a) \le \lambda\} \quad \text{and} \quad \#\{(n,m) : \, \mu_{nm}^{\mathbb{N}}(a) \le \lambda\},$$

summing them up and letting $a \to 0$ we proof the result.

Remark.

Note that the Dirichlet-Neumann bracketing can be applied only to domains with relatively smooth boundaries. One can extend Weyl's asymptotics to arbitrary domains of finite measure only if one has a uniform estimated for $N(\lambda, -\Delta_\Omega^D)$ with some constant C

$$N(\lambda, -\Delta_\Omega^D) \leq C\,|\Omega|\,\lambda^{d/2}.$$

- Weyl's conjecture.

In H.Weyl ('1911) also conjectured that

$$N(\lambda) = L_{0,d}^{cl}\,\lambda^{d/2}|\Omega| - c_{d-1}\,\lambda^{(d-1)/2}|\partial\Omega| + o(\lambda^{(d-1)/2}),$$

where $c_{d-1} > 0$ is a standard term depending only on dimension d.

Under certain conditions on classical billiards in $T^*\Omega$ V.Ivrii ('80) proved this result.

• Easier question.

Is there a constant $C \geq 1$ such that

$$N(\lambda) \leq C\, L_{0,d}^{cl}\, |\Omega|\, \lambda^{d/2} \ ?$$

$*$ This inequality was proved for bounded domains by Birman & Solomyak ('70) and Ciesielski ('70) with some constant $\tilde{C} > 1$.

$*$ For domains of finite measure and with some $\tilde{C} > 1$ it was proved by G.Rosenblum ('71) and E.Lieb ('80).

$*$ The best known constant is due to Li &Yau ('83).

- The Berezin–Li–Yau inequalities.

Theorem. (AL '93) Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure and $\lambda > 0$. Then

$$\sum_k (\lambda - \lambda_k)_+ \le (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \, d\xi$$

$$= \lambda^{1+d/2} \, |\Omega| \, (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+ \, d\xi.$$

Proof. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. Then the spectrum of $-\Delta_\Omega^D$ is discrete. We denote by λ_k the non-decreasing sequence of eigenvalues (counting with multiplicities) and let φ_k be an associated othonormal system of Dirichlet eigenfunctions. Note that these eigenfunctions can be continued by zero outside of Ω to H^1-functions on $\mathbb{R}^d$.

For any $\lambda > 0$ we have

$$\sum_k (\lambda - \lambda_k)_+ = \sum_k \left(\int_\Omega (\lambda |\varphi_k|^2 - |\nabla \varphi_k|^2) \, dx \right)_+$$

$$= \sum_k \left((2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2) \, |\widehat{\varphi_k}|^2) \, d\xi \right)_+ \le (2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \sum_k |\widehat{\varphi_k}|^2 \, d\xi.$$

Clearly

$$\sum_k |\widehat{\varphi_k}|^2 = \sum_k \left| (\varphi_k, e^{-ix\xi})_{L^2(\Omega)} \right|^2 = \|e^{-ix\xi}\|^2 = |\Omega|.$$

The proof is complete.

Corollary. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. Then for all $\lambda > 0$, we have

$$N(\lambda, -\Delta_\Omega^D) \leq \left(1 + \frac{2}{d}\right)^{d/2} L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}.$$

Proof. Consider

Here $\rho = \eta - \lambda$. Then by the BLY inequality we have

$$N(\lambda, -\Delta_\Omega^D) \leq \frac{1}{\eta - \lambda} \sum_k (\eta - \lambda_k)_+ \leq \frac{1}{\eta - \lambda} \, |\Omega| \, (2\pi)^{-d} \int_{\mathbb{R}^d} (\eta - |\xi|^2)_+ \, d\xi$$

$$= \frac{\eta^{1+d/2}}{\eta - \lambda} \, |\Omega| \, (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} (1 - |\xi|^2)_+ \, d\xi.$$

Minimising wrt η we find $\eta = \lambda \, \frac{1+d/2}{d/2}$ and obtain the proof.

Remark. Nobody knows if there is a constant $C_d : \ 1 \le C_d < \left(1 + \frac{2}{d}\right)^{d/2}$ such that

$$N(\lambda, -\Delta_\Omega^D) \le C_d \, L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}.$$

Remark. From Ivrii's asymptotics one can obtain

$$\begin{aligned}\mathrm{Tr}\,(-\Delta_\Omega^D - \lambda)_- &= \sum_k (\lambda - \lambda_k)_+ \\ &= L_{1,d}^{cl} \, |\Omega| \, \lambda^{1+d/2} - \frac{1}{4} \, L_{1,d-1}^{cl} \, |\partial\Omega| \, \lambda^{1+(d-1)/2} + o\left(\lambda^{1+(d-1)/2}\right).\end{aligned}$$

Some uniform wrt λ BLY inequalities with negative remainder terms were obtained in the papers of A.D.Melás; T.Weidl; L.Geisinger, T.Weidl and AL; L.Geisinger and T.Weidl.

- Exercises

$*$ Let $\Omega \subset \mathbb{R}^d$ be a domain of finite measure and let $-\Delta_\Omega^N$ be the Neumann Laplacian defined via its quadratic form

$$\int_\Omega |\nabla u|^2\, dx, \qquad u \in H^1(\Omega).$$

For domains with sufficiently 'good" boundaries the spectrum of $-\Delta_\Omega^N$ is discrete and consists of eigenvalues $\{\mu_k\}_{k=1}^\infty$ with $\mu_1 = 0$. It is enough to assume that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$,.

Prove Kröger's inequality ('92) that states

$$\sum_k (\mu - \mu_k)_+ \geq \mu^{1+d/2}\, |\Omega|\, (2\pi)^{-d} \int (1 - |\xi|^2)_+ d\xi.$$

$*$ Show that the BLY inequality for the Dirichlet Laplacian implies

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq \lambda^{\gamma+d/2}\, |\Omega|\, (2\pi)^{-d} \int (1 - |\xi|^2)_+^\gamma\, d\xi.$$

We usually use the notation

$$L_{\gamma,d}^{cl} = (2\pi)^{-d} \int (1 - |\xi|^2)_+^\gamma\, d\xi,$$

for any $\gamma > 1$.

- Pólya's inequality for tiling domains (Pólya '61).

Definition.

An open set $\Omega \subset \mathbb{R}^d$ is called tiling if there are countable families of orthogonal matrices $\{R_n\}$ and of vectors $\{a_n\}$ such that the sets

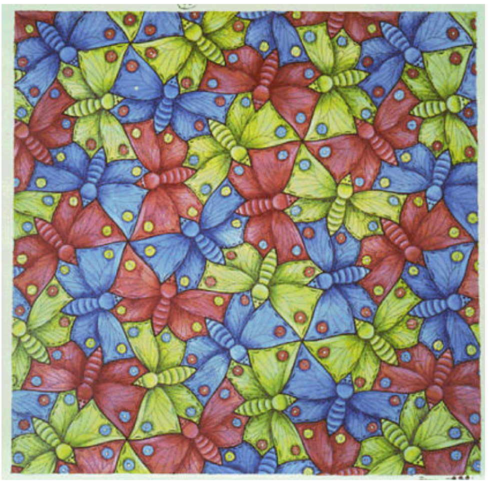
$$\Omega_n = \{R_n x + a_n : x \in \Omega\}$$

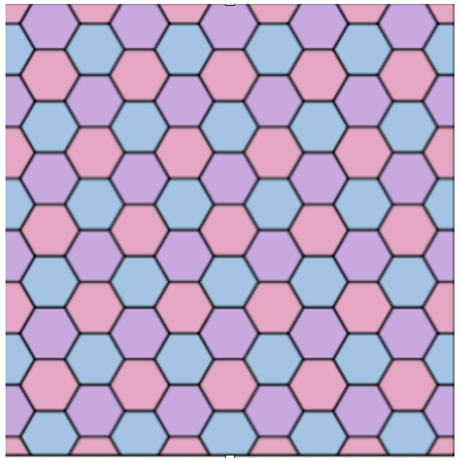
satisfy the following two properties

(1) $\Omega_n \cap \Omega_m = \emptyset$, if $n \neq m$,

(2) $\left|\mathbb{R}^d \setminus \cup \Omega_n\right| = 0$.

Of course, triangles, parallelograms, regular hexagons are examples of tiling domains but there are more exotic examples.





Theorem. (Pólya '61)

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set which is tiling. Then for any $\lambda > 0$,

$$N(\lambda, -\Delta_\Omega^D) \leq L_{0,d}^{cl} \, |\Omega| \, \lambda^{d/2}.$$

Proof.

For $L > 0$ we introduce the cube $Q_L = (-L/2, L/2)^d$ and the sets

$$J_L = \{n : \Omega_n \subset Q_L\} \qquad \Omega^L = \text{int} \left(\cup_{n \in J_L} \overline{\Omega_n} \right) \subset Q_L.$$

Moreover, by property (2) in the definition of the tiling domain

$$\lim_{L \to \infty} L^{-d} \, \# J_L = |\Omega|^{-1}.$$

We have the operator inequalities

$$-\Delta_{Q_L}^D \leq -\Delta_{\Omega^L}^D \leq \oplus_{n \in J_L} (-\Delta_{\Omega_n}^D).$$

Since all the domains Ω_n are congruent to Ω, the Laplacians $-\Delta^D_{\Omega_n}$ are unitarily equivalent to $-\Delta^D_{\Omega}$. Hence

$$N(\lambda, -\Delta^D_{Q_L}) \geq N\left(\lambda, \oplus_{n\in J_L}(-\Delta^D_{\Omega_n})\right) = (\#J_L)\, N(\lambda, -\Delta^D_{\Omega})$$

or

$$N(\lambda, -\Delta^D_{\Omega}) \leq \frac{L^d}{\#J_L}\,\frac{N(\lambda, -\Delta^D_{Q_L})}{L^d}.$$

By using scaling $x \to x/L$, note, that $-\Delta^D_{Q_L}$ is unitarily equivalent to the operator $-L^{-2}\Delta^D_{Q_1}$ in $L^2(Q_1)$. Hence

$$N(\lambda, -\Delta^D_{Q_L}) = N(L^2\,\lambda, -\Delta^D_{Q_1}).$$

Let us consider the Dirichlet Laplacian $-\Delta^D_{Q_1}$ in $L^2(Q_1)$. The eigenvalues and eigenfunctions of this operator are

$$\psi_n(x) = \prod_j \sin\left(\pi(x+1/2)n_j\right), \ x \in (-1/2, 1/2), \ \text{and } \lambda_n = \pi^2\,|n|^2,$$

where $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$ and $|n|^2 = n_1^2 + n_2^2 + \cdots + n_d^2$.

Then

$$\begin{aligned} N(\lambda, -\Delta^D_{Q_1}) &= \#\left\{n: \ |n|^2 < \lambda/\pi^2\right\} \\ &\sim \frac{1}{2^d}\int_{|\xi|^2<\lambda/\pi^2} d\xi = \frac{1}{(2\pi)^d}\,\lambda^{d/2}\int_{|\xi|<1} d\xi \\ &= \lambda^{d/2} L^{cl}_{0,d}, \qquad \text{as } \lambda \to \infty. \end{aligned}$$

This is the “Weyl” asymptotic law for cubes.

We now have as $L \to \infty$

$$\lim_{L\to\infty} L^{-d}\, N(\lambda, -\Delta^D_{Q_L}) = \lambda^{d/2} \lim_{L\to\infty} (L^2\lambda)^{-d/2}\, N(L^2\,\lambda, -\Delta^D_{Q_1}) = \lambda^{d/2} L^{cl}_{0,d}.$$

Thus, since $\lim_{L\to\infty} L^{-d}\,\#J_L = |\Omega|^{-1}$ we obtain

$$N(\lambda, -\Delta^D_\Omega) \le \frac{L^d}{\#J_L}\,\frac{N(\lambda, -\Delta^D_{Q_L})}{L^d} \to |\Omega|\,\lambda^{d/2} L^{cl}_{0,d}, \quad \text{as } L \to \infty.$$

Remark. Obviously the two dimensional disc is not a tiling domain.

Very recently M.Levitin, I.Polterovich and D.A.Sher have proved Pólya's conjecture for the unit disc. You can find it in:

https://arxiv.org/pdf/2203.07696.pdf

- Pólya's inequality for product domains.

Theorem. Let $d = d_1 + d_2$ with $d_1 \geq 2$ and $d_2 \geq 1$. Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be open sets of finite measure and assume that

$$N(\lambda, -\Delta_{\Omega_1}^D) \leq L_{0,d_1}^{cl}\, \lambda^{d_1/2} |\Omega_1|, \qquad \forall \lambda > 0.$$

Then for $\Omega = \Omega_1 \times \Omega_2$

$$N(\lambda, -\Delta_{\Omega}^D) \leq L_{0,d}^{cl}\, \lambda^{d/2} |\Omega|, \qquad \forall \lambda > 0.$$

Proof. Let $\{\lambda_n^{(j)}\}$, $n \in \mathbb{N}$, be the eigenvalues of $-\Delta_{\Omega_j}^D$. Then the eigenvalues of $-\Delta_{\Omega}^D$ are given by $\lambda_n^{(1)} + \lambda_m^{(2)}$, $(n,m) \in \mathbb{N} \times \mathbb{N}$.

Then

$$\begin{aligned} N(\lambda, -\Delta_{\Omega}^D) &= \# \left\{ (n,m) :\ \lambda_n^{(1)} + \lambda_m^{(2)} < \lambda \right\} \\ &= \sum_m \# \left\{ n \in \mathbb{N} :\ \lambda_n^{(1)} < \lambda - \lambda_m^{(2)} \right\} = \sum_m N\left(\lambda - \lambda_m^{(2)}, -\Delta_{\Omega_1}^D \right) \\ &\leq L_{0,d_1}^{cl} |\Omega_1| \sum_m (\lambda - \lambda_m)_+^{d_1/2} \leq L_{0,d_1}^{cl}\, L_{d_1/2,d_2}^{cl}\, |\Omega_1|\, |\Omega_2|\, \lambda^{(d_1+d_2)/2}, \end{aligned}$$

where

$$L_{\gamma,d}^{cl} = (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^{\gamma}\, d\xi.$$

It remains to show that

$$L^{cl}_{0,d} = L^{cl}_{0,d_1}\, L^{cl}_{d_1/2,d_2}.$$

Indeed

$$\begin{aligned} L^{cl}_{0,d} = (2\pi)^{-d} \left| \left\{ (\xi_1,\xi_2) : \ |\xi_1|^2 + |\xi_2|^2 < 1 \right\} \right| \\ (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} (2\pi)^{-d_1} \ \left| \left\{ \xi_1 : \ |\xi_1|^2 < (1 - |\xi_2|^2)_+ \right\} \right| \, d\xi_2 \\ = L^{cl}_{0,d_1}\, (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} (1 - |\xi_2|^2)_+^{d_1/2}\, d\xi_2 \\ = L^{cl}_{0,d_1}\, L^{cl}_{d_1/2,d_2}. \end{aligned}$$

Thank you