Spectral Theory and its Applications -
- Spectral and Functional Inequalities

A. Laptev

Imperial College London

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Lecture 3

• Comparing Dirichlet and Neumann eigenvalues.

Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure and let $-\Delta_\Omega^D$ and $-\Delta_\Omega^N$ be the Dirichlet and the Neumann Laplacians, respectively. As before we assume that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, which implies that the spectrum of both operators are discrete and accumulate at infinity.

The min-max principle implies

$$\mu_k(-\Delta_\Omega^N) \le \lambda_k(-\Delta_\Omega^D), \qquad \forall k \in \mathbb{N}.$$

Remark. If $d = 1$ and $\Omega = (0, \pi)$ then obviously

$$\mu_{k+1}(-\Delta_\Omega^N) \le \lambda_k(-\Delta_\Omega^D), \quad \forall k \in \mathbb{N}.$$

Theorem. (N.Filonov '04) If $d \ge 2$, then

$$\mu_{k+1}(-\Delta_\Omega^N) \le \lambda_k(-\Delta_\Omega^D), \qquad \forall k \in \mathbb{N}.$$

This theorem settles a question of Payne from 1955.

* Pólya (1952) proved it for $k = 1$.

* Payne (1955) obtained for two-dimensional convex domains

$$\mu_{k+2}(-\Delta_\Omega^N) \leq \lambda_k(-\Delta_\Omega^D), \qquad \forall k \in \mathbb{N}.$$

* Levine and Weinberger (1986) extended Payne's method to any dimension d and obtained

$$\mu_{k+r}(-\Delta_\Omega^N) \leq \lambda_k(-\Delta_\Omega^D), \qquad \forall k \in \mathbb{N}, \quad 1 < r \leq d.$$

if certain conditions (depending on r) on the principal curvatures of Ω are satisfied.

* Friedlander (1991) obtained for bounded domains with C^1-boundaries that

$$\mu_{k+1}(-\Delta_\Omega^N) \leq \lambda_k(-\Delta_\Omega^D), \qquad \forall k \in \mathbb{N}.$$

As a first step in the proof we note that a non-trivial eigenfunction of the Neumann Laplacian cannot satisfy Dirichlet boundary conditions.

Lemma. For any $\mu \geq 0$ we have

$$\ker\,(-\Delta_\Omega^N - \mu) \cap H_0^1(\Omega) = \{\emptyset\}.$$

Proof. I leave it as an exercise.

Proof of Theorem.

Let $L = P_{(-\infty,\mu]}L^2(\Omega)$, where P is the spectral measure of the Dirichlet Laplacian. Then $L \subset H_0^1(\Omega)$ and

$$\dim L = N(\mu, -\Delta_\Omega^D) + \dim\ker(-\Delta_\Omega^D - \mu) < \infty.$$

Moreover,

$$\int_\Omega |\nabla u|^2\,dx \le \mu \int_\Omega |u|^2\,dx, \quad \forall\, u \in L. \qquad (*)$$

By Lemma $\ker(-\Delta_\Omega^N - \mu)$ has only a trivial intersection with $H_0^1(\Omega)$ and therefore also with L. Hence the sum

$$F = L \dotplus \ker(-\Delta_\Omega^N - \mu)$$

is direct and finite-dimensional.

On the other hand, the linear span of the functions $e^{i\omega x}|_\Omega$, $\omega \in \mathbb{R}^d$ satisfying $|\omega| = \sqrt{\mu}$ is infinite-dimensional. Thus, there is at least one vector $\omega_0 \in \mathbb{R}^d$ with $|\omega_0| = \sqrt{\mu}$ such that $e^{i\omega_0 x} \notin F$. For the direct sum

$$G = L \dotplus \ker(-\Delta_\Omega^N - \mu) \dotplus \operatorname{span}\{e^{i\omega_0 x}\}$$

we have $G \subset H^1(\Omega)$ and

$$\dim G = \dim L + \dim\ker(-\Delta_\Omega^N - \mu) + 1.$$

Let us show that inequality $(*)$ remains true for all $u \in G$. Indeed, for any $u = u_L + v + ce^{i\omega_0 x}$, $u_L \in L$, $v \in \ker(-\Delta_\Omega^N - \mu)$, $c \in \mathbb{C}$, we have

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega |\nabla u_L + \nabla v + c\nabla e^{i\omega_0 x}|^2 \, dx = I_1 + I_2.$$

Here

$$\begin{aligned} I_1 = \int_\Omega \left(|\nabla u_L|^2 + |\nabla v|^2 + |\nabla c\, e^{i\omega_0 x}|^2\right) \, dx \\ \leq \mu \int_\Omega \left(|u_L|^2 + |v|^2 + |ce^{i\omega_0 x}|^2\right) \, dx \end{aligned}$$

and

$$I_2 = 2\mathrm{Re} \int_\Omega \left(\nabla v \cdot \overline{\nabla(u_L + ce^{i\omega_0 x})} + c\nabla e^{i\omega_0 x} \cdot \overline{\nabla u_L}\right) \, dx$$

Since the eigenfunction v belongs to the operator domain of $-\Delta_\Omega^N$ and $u_L + ce^{i\omega_0 x}$ belongs to the form domain $H^1(\Omega)$ of $-\Delta_\Omega^N$ and since $u_L \in H_0^1(\Omega)$, we have

$$I_2 = 2\mathrm{Re} \int_\Omega \Big((-\Delta_\Omega^N v)\, \overline{(u_L + ce^{i\omega_0 x})} - c\,(\Delta e^{i\omega_0 x})\, \overline{u_L} \Big)\, dx$$

$$= 2\mu\, \mathrm{Re} \int_\Omega \Big(v\overline{(u_L + ce^{i\omega_0 x})} + c\, e^{i\omega_0 x}\, \overline{u_L} \Big)\, dx.$$

Thus

$$\int_\Omega |\nabla u|^2\, dx \le \mu \int_\Omega \left(|u_L|^2 + |v|^2 + |ce^{i\omega_0 x}|^2 \right)\, dx + I_2$$

$$= \mu \int_\Omega |u_L + v + ce^{i\omega_0 x}|^2\, dx.$$

and we obtain

$$N(\mu, -\Delta_\Omega^N) \ge \dim G = N(\mu, -\Delta_\Omega^D) + \dim\ker\,(-\Delta_\Omega^N - \mu) + 1.$$

If now we let now $\mu = \lambda_k$ we conclude that $\mu_{k+1} < \lambda_k$.

- Schrödinger operator.

Let us define $-\Delta - V$ as a self-adjoint operator in $L^2(\mathbb{R}^d)$ using the theory of quadratic forms

$$(\mathcal{H}u, u) = \int_{\mathbb{R}^d} (|\nabla u|^2 - V\,|u|^2)\,dx.$$

Proposition. Let $V \in L^1_{loc}(\mathbb{R}^d)$ be a real function on $\mathbb{R}^d$ and assume that there are constants $\theta < 1$ and $C < \infty$ such that

$$\int_{\mathbb{R}^d} V_+ |u|^2\,dx \le \theta \int_{\mathbb{R}^d} \left(|\nabla u|^2 + V_- |u|^2\right)\,dx + C \int_{\mathbb{R}^d} |u|^2\,dx \qquad \forall u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V_- dx).$$

Then the quadratic form

$$\int_{\mathbb{R}^d} \left(|\nabla u|^2 - V|u|^2\right)\,dx$$

with domain $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V_- dx)$ is closed and lower-semibounded in $L^2(\mathbb{R}^d)$ and $C_0^\infty(\mathbb{R}^d)$ is a form core.

Corollary. Let V be as in Proposition. Then there is a unique, lower semi-bounded and self-adjoint operator $\mathcal{H}$ in $L^2(\mathbb{R}^d)$ whose

$$\operatorname{dom}\mathcal{H} \subset H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V_- dx)$$

and for all $u \in \operatorname{dom}\mathcal{H}$ and $v \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V_- dx)$

$$\int_{\mathbb{R}^d} (\nabla u \cdot \nabla \overline{v} - V\, u\, \overline{v})\, dx = (\mathcal{H}u, v).$$

Its domain is given by

$$\operatorname{dom}\mathcal{H} = \{u \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V_- dx) :\ -\Delta u - V\, u \in L^2(\mathbb{R}^d)\}.$$

For $u \in \operatorname{dom}\mathcal{H}$ we have

$$\mathcal{H}u = -\Delta u - V\, u.$$

Proposition. Let V be a real-valued function on $\mathbb{R}^d$ such that $V_- \in L^1_{loc}(\mathbb{R}^d)$ and $V_+ \in L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$, where

$$\begin{cases} p = 1 & if\, d = 1 \\ p > 1 & if\, d = 2 \\ p = d/2 & if\, d \geq 3. \end{cases}$$

Then for any $\theta > 0$ there is $C > 0$ such that for any $u \in H^1(\mathbb{R}^d)$

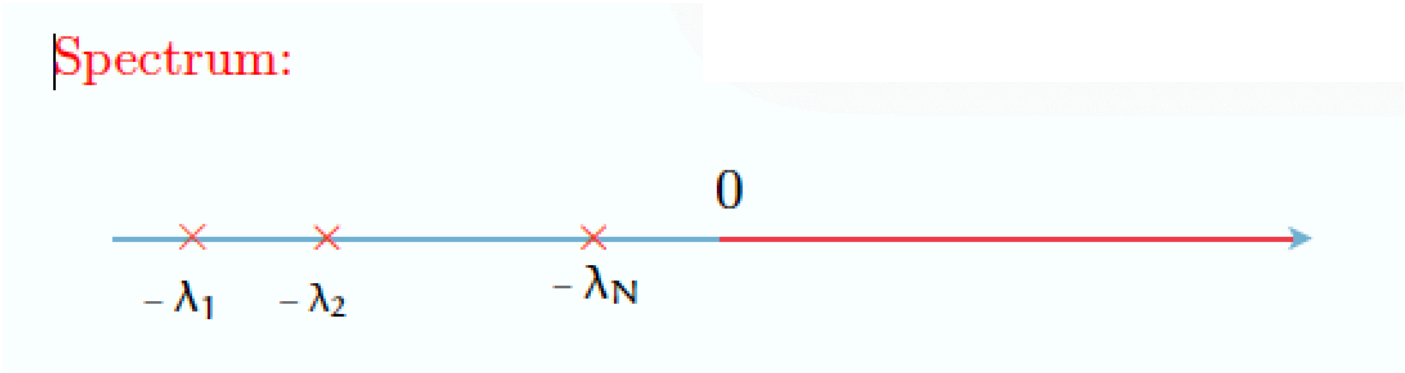
$$\int_{\mathbb{R}^d} V_+ |u|^2 \, dx \leq \theta \int_{\mathbb{R}^d} \left(|\nabla u|^2 + V_- |u|^2 \right) \, dx + C \int_{\mathbb{R}^d} |u|^2 \, dx.$$

- CLR and Lieb-Thirring inequalities

Consider a 1D Schrödinger operator

$$\mathcal{H} = -\frac{d^2}{dx^2} - V(x), \qquad \text{in } L^2(\mathbb{R}),$$

where $V \to 0$ as $|x| \to \infty$ and $V \geq 0$. and let $\{-\lambda_k\}$ be negative eigenvalues of $\mathcal{H}$.



$$\sum_j |\lambda_j|^\gamma = \sum_j |\lambda_j(V)|^\gamma \leq \frac{C_{d,\gamma}}{(2\pi)^d} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma dx d\xi = L_{\gamma,d} \int V(x)_+^{\gamma+d/2}\, dx.$$

This inequality holds true for $d = 1$, $\gamma \geq 1/2$; $d = 2$, $\gamma > 0$; $d \geq 3$, $\gamma \geq 0$.

Compare with Weyl's asymptotic formula:

$$\sum_j |\lambda_j(\alpha V)|^\gamma \sim_{\alpha\to\infty} L^{cl}_{\gamma,d} \int (\alpha V_+)^{\gamma+d/2}\, dx = (2\pi)^{-d} \iint (\xi^2 - \alpha V)_-^\gamma \, d\xi dx,$$

which implies $L^{cl}_{\gamma,d} \leq L_{\gamma,d}$.

Applications.

- Weyl's asymptotics.
- Stability of matter.
- Study of properties of continuous spectrum of Schrödinger operators.
- Estimate of dimensions of attractors in theory of Navier-Stokes equations.
- Bounds on the maximum ionization of atoms.

$$\sum_j |\lambda_j(V)|^\gamma \leq \frac{C_{d,\gamma}}{(2\pi)^d} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi.$$

Remark.

If in $H = -\Delta + V$,

$$V(x) = \begin{cases} -\lambda, & x \in \Omega, \\ +\infty, & x \notin \Omega, \end{cases} \qquad \Omega \in \mathbb{R}^d,$$

then the spectrum of H coincides with the spectrum of the Dirichlet Laplacian in Ω.

Therefore Pólya inequalities are special cases of L-Th inequalities.

- CLR and Lieb-Thirring inequalities.

Theorem. Let $\gamma > 1/2$ if $d = 1$, $\gamma > 0$ if $d \geq 2$ and let $0 \leq V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then the negative eigenvalues $\{-\lambda_k\}$ of the operator $-\Delta - V$ satisfy

$$\sum_k \lambda_k^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+d/2}\, dx.$$

Sharp constant were obtained in the following cases:

Theorem. It is known that $L_{1/2,1} = 1/2$ ($L^{cl}_{1/2,1} = 1/4$) and
$L_{\gamma,d} = L^{cl}_{\gamma,d}$ if $\gamma \geq 3/2$, $d \geq 1$.
In other cases the sharp constants are unknown.

E.Lieb, W.Thirring, M.Aizenmann, D.Hundertmark, L.Thomas, AL & T.Weidl.

- Darboux transform $\gamma = 3/2$.

Let $(-\lambda_1, \psi_1)$ be the lowest eigenvalue and its respective eigenfunction

$$\mathcal{H}\psi_1 = -\frac{d^2}{dx^2}\psi_1 - V(x)\,\psi_1 = -\lambda_1\,\psi_1.$$

It is known that $\psi_1 \neq 0$ and we can choose $\psi_1 > 0$.

Denote

$$f_1 = \frac{\psi_1'}{\psi_1}, \qquad f_1' = \frac{\psi_1''}{\psi_1} - \left(\frac{\psi_1'}{\psi_1}\right)^2.$$

Therefore

$$f_1' + f_1^2 = \frac{\psi_1''}{\psi_1} = \lambda_1 - V.$$

Let us introduce

$$Q_1 = \frac{d}{dx} - f_1 \qquad \& \qquad Q_1^* = -\frac{d}{dx} - f_1.$$

Then

$$Q_1^* Q_1 = \left(-\frac{d}{dx} - f_1\right)\left(\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} + f_1' + f_1^2$$
$$= -\frac{d^2}{dx^2} - V + \lambda_1 = \mathcal{H} + \lambda_1.$$

The discrete spectrum $\sigma_d(Q_1^* Q_1)$ of the operator $Q_1^* Q_1$ coincides with

$$\sigma_d(Q_1^* Q_1) = \{0, -\lambda_2 + \lambda_1,\ -\lambda_3 + \lambda_1, \dots\}.$$

In particular,

$$Q_1^* Q_1 \psi_1 = 0,$$

where

$$\psi_1(x) \sim \begin{cases} e^{-\sqrt{\lambda_1} x}, & x \to +\infty, \\ e^{\sqrt{\lambda_1} x}, & x \to -\infty. \end{cases}$$

and also

$$f_1(x) = \frac{\psi_1'(x)}{\psi_1(x)} \sim \begin{cases} -\sqrt{\lambda_1}, & x \to +\infty, \\ \sqrt{\lambda_1}, & x \to -\infty. \end{cases}$$

Commuting Q_1^* and Q_1 we obtain

$$Q_1Q_1^* = \left(\frac{d}{dx} - f_1\right)\left(-\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} - f_1' + f_1^2$$
$$= -\frac{d^2}{dx^2} - 2f_1' - V + \lambda_1 = \mathcal{H} - 2f_1' + \lambda_1.$$

The operators $Q_1^*Q_1$ and $Q_1Q_1^*$ have the same non-zero spectrum.

Moreover, $0 \notin \sigma(Q_1Q_1^*)$, indeed, assume that there is $\psi \in L^2(\mathbb{R})$ s.t.

$$Q_1Q_1^*\psi = 0 \quad \Longrightarrow \quad \|Q_1^*\psi\| = 0 \quad \Longrightarrow$$
$$-\psi' - f_1\psi = 0 \quad \Longrightarrow \quad (f_1 \sim -\sqrt{\lambda_1},\, x \to +\infty) \quad \Longrightarrow$$
$$\psi \sim e^{\sqrt{\lambda_1}\, x}, \quad x \to +\infty.$$

Therefore $\psi \notin L^2(\mathbb{R})$.

Conclusion:

$$\sigma_d(\mathcal{H}) = \{-\lambda_1, -\lambda_2, -\lambda_3, \dots\}$$

and

$$\sigma_d(\mathcal{H} - 2f_1') = \{-\lambda_2, -\lambda_3, \dots\}.$$

Denote now $V_1 = V + 2f_1'$, $\mathcal{H}_1 = \mathcal{H} - 2f_1'$.

Considering the class of potentials with the finite number of eigenvalues and repeating this process, we obtain a non-negative Schrödinger operator with the potential

$$-V_n = -V - 2f_1' - 2f_2' - \cdots - 2f_n',$$

where

$$f_n' + f_n^2 = \lambda_n - V_{n-1}$$

and

$$\sigma_d(\mathcal{H} - 2f_1' - 2f_2' - \cdots - 2f_n') = \emptyset.$$

Finally we have

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}} (V_{n-1} + 2f_n')^2 \, dx = \int_{\mathbb{R}} \left[V_{n-1}^2 + 4f_n'(V_{n-1} + f_n') \right] \, dx \\
&= \int_{\mathbb{R}} \left[V_{n-1}^2 + 4f_n'(\lambda_n - f_n^2) \right] \, dx = \int_{\mathbb{R}} V_{n-1}^2 \, dx + 4\lambda_n f_n \Big|_{-\infty}^{\infty} - \frac{4}{3} f_n^3 \Big|_{-\infty}^{\infty} \\
&= \int_{\mathbb{R}} V_{n-1}^2 \, dx - 8\lambda_n^{3/2} + \frac{8}{3} \lambda_n^{3/2} = \int_{\mathbb{R}} V_{n-1}^2 \, dx - \frac{16}{3} \lambda_n^{3/2} \\
&= \cdots = \int_{\mathbb{R}} V^2 \, dx - \frac{16}{3} \sum_{j=1}^{n} \lambda_j^{3/2}.
\end{aligned}$$

Theorem. (Benguria and Loss '00)

Let $\mathcal{H} = -\frac{d^2}{dx^2} - V$, in $L^2(\mathbb{R})$, where $V \in L^2(\mathbb{R})$, $V \geq 0$. Then for the negative eigenvalues $\{\lambda_j\}$ of the operator $\mathcal{H}$ we have

$$\sum_j \lambda^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} V^2 \, dx.$$

Remark. The above inequality holds for $H = -\frac{d^2}{dx^2} - V$ in $L^2(\mathbb{R}_+)$ with Dirichlet boundary conditions at zero.

• Sharp multidimensional Lieb-Thirring inequalities, $\gamma = 3/2$.

The main argument is based on a Lieb-Thirring inequality for Schrödinger operators with matrix-valued potentials.

Theorem. (AL & T.Weidl)
Let $M \geq 0$ be a Hermitian $m \times m$ matrix-function and let $\mathcal{H} = -d^2/dx^2 - M$ in $L^2(\mathbb{R})$. Then

$$\sum_n \lambda_n^{3/2}(\mathcal{H}) \leq \frac{3}{16} \int \mathrm{Tr}\, M^2(x)\, dx.$$

Using this result we prove the following we obtain

Theorem. (AL & T.Weidl)
Let $V \in L^\gamma(\mathbb{R}^d)$ with $\gamma \geq 3/2$. Then for the negative eigenvalues of the Schrödinger operator

$$\mathcal{H} = -\Delta - V$$

we have

$$\sum |\lambda_n^\gamma \leq L_{\gamma,d}^{cl} \int_{\mathbb{R}^d} V^{\gamma+d/2}(x)\, dx.$$

For the proof we use the so-called lifting argument with respect to dimension. Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $x = (x_1, x_2)$ and $\gamma = 3/2$. Then

$$H = -\Delta - V = -\partial^2_{x_1 x_1} - \underbrace{(\partial^2_{x_2 x_2} + V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$. Thus $\tilde{H}_+(x_1)$ has a finite rank. Let, for instance, $\gamma = 3/2$

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1 x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16} \int \operatorname{Tr} \tilde{H}_+^2(x_1)\, dx_1 \leq \underbrace{\frac{3}{16} L_{2,1}}_{L^{cl}_{3/2,2}} \iint V^{3/2+1}(x)\, dx.$$

Thank you