Spectral Theory and its Applications -
- Spectral and Functional Inequalities

A. Laptev

Imperial College London

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Lecture 5

- Calogero inequality.

Let $V \geq 0$, $V \in L^{d/2}(\mathbb{R}^d)$, and let H be a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$Hu = -\Delta u - V\,u.$$

The operator H might have finite or infinite number of negative eigenvalues λ_k. Denote by $N(H)$ the number of such eigenvalues

$$N(H) = \#\{k:\ \lambda_k < 0\}.$$

Then if $d \geq 3$ the CLR inequality provides an upper bound

$$N(H) \leq C_d \int_{\mathbb{R}^d} V^{d/2}(x)\,dx.$$

It is well known that $d = 1$ or 2 the latter inequality does not hold for arbitrary $V \in L^{d/2}(\mathbb{R}^d)$.

If $d = 1$ there is a classical Calogero inequality:

Let H be a Schrödinger operator in $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, with the Dirichlet boundary condition

$$Hu = -u'' - Vu, \quad \text{where} \quad u(0) = 0.$$

Theorem. (Calogero 1965)

Assume that $V \geq 0$, $V \in L^{1/2}(\mathbb{R}_+)$, is monotonically decaying function. Then for the number $N(H)$ of the negative eigenvalues for the operator H we have

$$N(H) \leq \frac{2}{\pi} \int_0^\infty \sqrt{V(x)}\, dx = 2\, \frac{1}{2\pi} \int_{\mathbb{R}^2} (\xi^2 - V(x))_-^0\, d\xi dx.$$

Remarks.

1. Note that if $u \in H_0^1(\mathbb{R}_+)$ we have Hardy's inequality

$$\int_0^\infty |u'(x)|^2\,dx \geq \frac{1}{4}\int_0^\infty \frac{u^2(x)}{x^2}\,dx.$$

2. The multi-dimensional Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx$$

makes sense for $d \geq 3$.

This leads us to the fact that the existence of semiclassical CLR inequalities are connected on Hardy type inequalities.

One of the ideas was to simply add a Hardy term (AL '00).

Let us consider a Schrödinger operator in $L^2(\mathbb{R}^2)$

$$H_\beta = -\Delta + \frac{\beta}{|x|^2} - V,$$

where $\beta > 0$, $V \geq 0$, $V \in L^1(\mathbb{R}^2)$ and $V(x) = V(|x|)$. Then

$$N(H_\beta) \leq C_\beta \int_{\mathbb{R}^2} V(x)\, dx$$

where

$$C_\beta = (4\pi)^{-1} \sup_{\mu > 0} \left\{ \mu^{-1/2} \cdot \#\{n \in \mathbb{Z} : \, n^2 + \beta - \mu < 0\} \right\}.$$

Remark. The constant C_α is sharp and $C_\alpha \to \infty$ as $\alpha \to 0$.

Proof. Introducing polar coordinates $x = (r, \theta)$ and changing variables $r = e^t$ we reduce the problem to the study of number of negative eigenvalues for the operator in $L^2(\mathbb{R} \times \mathbb{S}^1)$

$$-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \theta^2} + \beta - e^{2t}\, V(e^t).$$

Let $\{-\mu_k\}$ be negative eigenvalues of the operator $d^2/dt^2 - e^{2t}\, V(e^t)$. Then

$$\begin{aligned} N(H_\beta) &= \#\{k, n : \, n^2 - \mu_k + \beta < 0\} = \sum_k \sum_{n \in \mathbb{Z}:\, n^2 + \beta < \mu_k} 1 \\ &\leq 4\pi C_\beta \sum_k \sqrt{\mu_k} \leq C_\beta \int_{\mathbb{R}} e^{2t}\, V(e^t)\, dt = C_\beta \int_{\mathbb{R}^2} V(x)\, dx. \end{aligned}$$

• Magnetic 2D Schrödinger operators.

There is a magnetic 2D Hardy's inequality obtained in L-Weidl '99. Namely if we have a magnetic Laplacian with Aharonov-Bohm magnetic field

$$A = (A_1, A_2) = \alpha \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \qquad (x_1, x_2) \in \mathbb{R}^2.$$

Then, see Lecture 1,

$$\int_{\mathbb{R}^2} |(i\nabla + A)u|^2 \, dx \ge \psi_\alpha \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx,$$

where $\psi_\alpha = \min_{k\in\mathbb{Z}} |\alpha - k|^2$.

Denoting by $H_A - V = (i\nabla + A)^2 - V$ with a potential $V \in L^1(\mathbb{R}_+, r\,dr; L^\infty(\mathbb{S}^1))$ and using the latter Hardy inequality Balinsky-Evans-Lewis '01 obtained

$$N(H_A - V) \le C_\alpha \int_{\mathbb{R}_+} \|V(r,\cdot)\|_{L^\infty(\mathbb{S}^1)} r \, dr \, .$$

Remark. During the last two decades there were a number of deep results regarding CLR inequalities for 2D Schrödinger operators in the paper Solomyak, L-Netrusov, AL-Solomyak, Grigoryan-Nadirashvilli and E.Shargorodsky.

One of our main results can be seen as a generalisation of the Calogero bound to dimension $d = 2$. Let

$$\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1 > 0,\ x_2 \in \mathbb{R}\}.$$

Theorem. Let $V \in L^1(\mathbb{R}^2_+)$, $V \geq 0$, be non-increasing in x_1,

$$V(x_1, x_2) \geq V(x_1', x_2) \quad \text{for} \quad x_1 \leq x_1'.$$

Then the number of negative eigenvalues of the operator

$$H_0 - V = -\Delta - V$$

with Dirichlet boundary condition on $x_1 = 0$ satisfies the inequality

$$N(H_0 - V) \leq C \int_{\mathbb{R}^2_+} V(x)\, dx,$$

where the constant $C \leq 4.31$ is independent of V.

One of the main ingredients of the proof is a Calogero inequality with an operator-valued potential.

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{V}(t)$, $t \geq 0$, be an operator-valued function (potential), whose values are compact, non-negative self-adjoint operators in $\mathcal{H}$.
Let $\beta > 0$ and consider the operator $\mathcal{A}_\beta$ in $L^2(\mathbb{R}_+; \mathcal{H})$ with the Dirichlet boundary conditions whose quadratic form is

$$\mathcal{A}_\beta U(t) = -\frac{d^2}{dt^2} U(t) + \frac{\beta}{t^2} U(t) - \mathcal{V}(t)\, U(t), \qquad U(0) = 0.$$

We say that the family of operators $\mathcal{V}(t)$ is non-increasing if $\mathcal{V}(t) \geq \mathcal{V}(s)$ for $0 \leq t < s$ in the usual sense of quadratic forms. Namely, for any fixed vector $U \in \mathcal{H}$

$$(\mathcal{V}(t)U, U)_{\mathcal{H}} \geq (\mathcal{V}(s)U, U)_{\mathcal{H}}, \qquad 0 \leq t < s,$$

where $(\cdot\,, \cdot)_{\mathcal{H}}$ is the scalar product in $\mathcal{H}$.
Denote by $\{\mu_n(t)\}$ the eigenvalues of the self-adjoint operator $\mathcal{V}(t)$.

Proposition 1. Let the operator-valued potential $\mathcal{V} \geq 0$ is non-increasing. Then the number of the negative eigenvalues $N(\mathcal{A}_\beta)$ of the operator $\mathcal{A}_\beta$ satisfies the inequality

$$N(\mathcal{A}_\beta) \leq C_\beta \int_{\mathbb{R}_+} \left(\sum_n \sqrt{\mu_n(t)} \right) dt,$$

where C_β is a constant independent of $\mathcal{V}$.

Proof. We divide the semi-axis $\mathbb{R}_+$ into intervals $(2^k, 2^{k+1})$, $k \in \mathbb{Z}$. Denote by $\mathcal{A}_{\beta,k}$ the operator $\mathcal{A}_\beta$ restricted to the interval $(2^k, 2^{k+1})$ with Neumann boundary conditions. Then

$$N(\mathcal{A}_\beta) \leq \sum_{k \in \mathbb{Z}} N(\mathcal{A}_{\beta,k}).$$

Since $\mathcal{V}$ is monotone we have $\mathcal{V}(2^k) \geq \mathcal{V}(t)$, $t \in (2^k, 2^{k+1})$ and therefore

$$N(\mathcal{A}_\beta) \leq \sum_{k \in \mathbb{Z}} N \left(-\frac{d^2}{dt^2} + \frac{\beta}{t^2} - \mathcal{V}(2^k) \right).$$

By using scaling $t = 2^k s$ we reduce the problem on each of the intervals $(2^k, 2^{k+1})$ to the interval $(1, 2)$ and obtain

$$N(\mathcal{A}_\beta) \le \sum_{k\in\mathbb{Z}} N\left(-\frac{d^2}{ds^2} + \frac{\beta}{s^2} - 2^{2k}\,\mathcal{V}(2^k)\right)$$
$$\le \sum_{k\in\mathbb{Z}} N\left(-\frac{d^2}{ds^2} + \frac{\beta}{4} - 2^{2k}\,\mathcal{V}(2^k)\right).$$

We now consider the operator with constant coefficients

$$-\frac{d^2}{ds^2} + \beta/4 - 2^{2k}\,\mathcal{V}(2^k)$$

in the eigen-basis of the operator $\mathcal{V}(2^k)$. Denoting by $\mu_n(2^k)$ the eigenvalues of the operator $\mathcal{V}(2^k)$ we find

$$N(\mathcal{A}_\beta) \le \sum_{k,n} \left(\#\{m \ge 0 : \, m^2\pi^2/4 + \beta/4 - 2^{2k}\mu_n(2^k) < 0\}\right)$$
$$\le C \sum_{k,n} 2^k \sqrt{\mu_n(2^k)} \le C \int_{\mathbb{R}_+} \left(\sum_n \sqrt{\mu_n(t)}\right) dt.$$

Here we have used again the monotonicity of the eigenvalues of $V(t)$.

In order to prove our statement we consider the operator in $L^2(\mathbb{R}^2_+)$

$$H = H_0 - V = -\Delta - V$$

with Dirichlet boundary condition on $x_1 = 0$ and monotonically decaying wrt x_1 potential.
It can be written as

$$-\frac{d^2}{dx_1^2} - W(x_1),$$

where

$$W(x_1) = \frac{d^2}{dx_2^2} + V(x_1, x_2)$$

is an operator valued potential in $L^2(\mathbb{R})$.

Since functions from the domain of the operator H satisfy the Dirichlet boundary conditions at $x_1 = 0$ we use the 1D Hardy inequality and obtain

$$-\frac{d^2}{dx_1^2} = -\frac{1}{2}\frac{d^2}{dx_1^2} - \frac{1}{2}\frac{d^2}{dx_1^2} \geq -\frac{1}{2}\frac{d^2}{dx_1^2} + \frac{1}{8}\frac{1}{x_1^2}$$

and reduce the problem to the study of the operator

$$-\frac{1}{2}\frac{d^2}{dx_1^2} + \frac{1}{8}\frac{1}{x_1^2} - W(x_1).$$

Due to the variational principle we have

$$N\left(-\frac{1}{2}\frac{d^2}{dx_1^2} + \frac{1}{8}\frac{1}{x_1^2} - W(x_1)\right) \leq N\left(-\frac{1}{2}\frac{d^2}{dx_1^2} + \frac{1}{8}\frac{1}{x_1^2} - W(x_1)_+\right).$$

The operator $W(x_1)_+$ has discrete spectrum whose eigenvalues $\mu_n(x_1)$ satisfy the well-known sharp Lieb-Thirring inequalities

$$\sum_n \sqrt{\mu_n(x_1)} \leq \frac{1}{2}\int V(x_1, x_2)dx_2.$$

Applying Proposition 1 we finally obtain

$$N(H) \leq C \int_{\mathbb{R}^2_+} V(x_1, x_2)\, dx_1 dx_2.$$

• Magnetic 2D Schrödinger operators with non-increasing potentials.

Let us again consider the AB vector potential

$$A = (A_1, A_2) = \alpha \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \qquad (x_1, x_2) \in \mathbb{R}^2$$

and let $H_A - V = (i\nabla + A)^2 - V$.

We also have the following:

Theorem. Let $V \in L^1(\mathbb{R}^2), V \geq 0$ be a potential that is non-increasing along any ray from the origin, i.e. in polar coordinates $V(r,\varphi) \geq V(r',\varphi)$ for $r \leq r'$. If $\alpha \notin \mathbb{Z}$ then the number of negative eigenvalues of the operator $H_A - V$ satisfies the inequality

$$N(H_A - V) \leq C_\alpha \int_{\mathbb{R}^2} V(x)\, dx.$$

where the constant C_α is independent of V.

As before we need an appropriate Hardy inequality, a 1D Calogero inequality for operator-valued potentials, and Lieb-Thirring inequalities in this case for operators on $\mathbb{S}^1$.

Let $A = \alpha\left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$, $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ and consider the Aharonov-Bohm magnetic quadratic form whose respective operator is H_A in polar coordinates

$$\int_{\mathbb{R}^2} |(i\,\nabla + A)u|^2\, dx = \int_0^\infty \int_{\mathbb{S}^1} \left(|\partial_r u|^2 + \frac{|i\,\partial_\varphi u + \alpha)u|^2}{r^2}\right) r d\varphi dr.$$

Splitting $H_A = H_A/2 + H_A/2$, and using magnetic Hardy inequality we reduce the problem to the study the Schrödinger operator

$$H_A - V \geq \frac{1}{2}\, H_A + \frac{\psi_\alpha}{2}\, \frac{1}{|x|^2} - V.$$

Eventully we reduce the problem

$$\frac{1}{2}\,\frac{1}{r}\,\frac{d}{dr} r \frac{d}{dr} + \frac{\psi_\alpha}{2}\, \frac{1}{|x|^2} - W(r),$$

where

$$W(r) = -\frac{1}{r^2}\, \left((i\partial_\varphi + \alpha)^2 - r^2 V(r,\varphi)\right)$$

Proposition 2. Let us consider the magnetic operator on $L^2(\mathbb{S}^1)$

$$h_\alpha - v = (i\partial_\varphi + \alpha)^2 - v.$$

Then for the negative eigenvalues $-\mu_n$ we have

$$\sum_n \sqrt{\mu_n} \leq C_\alpha \int_{\mathbb{S}^1} v(\varphi)\, d\varphi.$$

Our proof follows the proof of T.Weidl '96 who obtained Lieb–Thirring inequality for the 1/2 moments of a Schrödinger operator on $\mathbb{R}$.

The main idea of the proof of Weidl is to use Neumann bracketing whereby $\mathbb{R}$ is partitioned into disjoint intervals that each support at most one eigenvalue. Importantly the length of each interval (compared to the L^1-norm of the potential on the interval) can be chosen to be uniformly bounded from below. To achieve the latter on the bounded set $(-\pi, \pi)$ we will use that for $\alpha \notin \mathbb{Z}$ the operator h does not have any zero-modes and that $h_\alpha - v$ does not admit any negative eigenvalues if

$$\int_{-\pi}^{\pi} v(\varphi)\, d\varphi$$

is small. The desired partition can then be constructed with multiplicity 2.

• Proof of the classical Calogero inequality.

Theorem. (Calogero 1965)

Assume that $V \geq 0$ is monotonically decaying function such that $V \in L^{1/2}(\mathbb{R}_+)$. Then for the number $N(H)$ of the negative eigenvalues for the operator $H = d^2/dx^2 - V$ we have

$$N(H) \leq \frac{2}{\pi} \int_0^\infty \sqrt{V(x)}\, dx.$$

Proof. It is enough to prove the theorem for smooth V's.
Let u be the solution of the equation on $\mathbb{R}_+$

$$-u'' - Vu = 0 \qquad u(0) = 0 \quad u'(0) = 1.$$

Assuming that $u'(0) \neq 0$ we introduce g satisfying the equation

$$\tan(g(x)) = V^{1/2}(x)\, \frac{u(x)}{u'(x)}.$$

Due to Sturm's oscillating theorem zeros of u that we denote by $\{z_k\}$ are simple as well as zeros of u' that we denote by $\{p_k\}$. We also have $z_k < p_k$ and

$$\mathrm{Im}\ \tan(g)(z_k, p_k) = (0, \infty)$$

and thus $g(z_k) = 0$ and $g(p_k) = \pi/2$.

Derivating both sides of the equation $\tan(g(x)) = V^{1/2}(x)\,\frac{u(x)}{u'(x)}$ and using that $u'' = -Vu$ we find

$$\frac{1}{\cos^2(g)}\,g' = \frac{1}{2}\,V'V^{-1/2}\frac{u}{u'} + V^{1/2} + V^{3/2}\left(\frac{u}{u'}\right)^2$$
$$= \frac{1}{2}\frac{V'}{V}\tan(g) + V^{1/2}\left(1+\tan^2(g)\right) = V^{1/2}\frac{1}{\cos^2(g)} + \frac{1}{2}\frac{V'}{V}\tan(g).$$

This implies

$$g' = V^{1/2} + \frac{1}{2}\frac{V'}{V}\tan(g)\cos^2(g) = V^{1/2} + \frac{1}{4}\frac{V'}{V}\sin(2g).$$

Since $g(z_k) = 0$ and $g(p_k) = \pi/2$ and since V is decaying (and thus $V' \le 0$)

$$\frac{V'}{V}\sin(2g) \le 0.$$

Finally we obtain

$$\int_{\mathbb{R}_+} V^{1/2}\,dx \ge \int_{z_k}^{p_k} \left(V^{1/2} + \frac{1}{2}\frac{V'}{V}\tan(g)\cos^2(g) \right) dx$$
$$= \int_{z_k}^{p_k} g'(x)\,dx = g(x)\Big|_{z_k}^{p_k} = \frac{\pi}{2}\,N(V),$$

where we used that the number of the negative eigenvalues $N(V)$ coincide with the number of intervals (z_k, p_k).

Thank you