Spectral Theory and its Applications -
- Spectral and Functional Inequalities

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Lecture 6

• Harmonic Oscillator.

Let us begin with the operator of harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R}.$$

The spectrum of this operator is discrete and equals $\{2k+1\}$, $k = 0, 1, 2, \ldots$. In particular,

$$H - 1 := A^* A = \left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right) \geq 0$$

which implies $H \geq 1$.

Remark. Note that the values of the symbol of the Harmonic oscillator $\xi^2 + x^2 \geq 0$ fills the semiaxis $[0, \infty)$ but the operator $H \geq 1$. The latter inequality is sharp and the eigenfunction corresponding to the eigenvalue one is $e^{-x^2/2}$.

Remark. It $\mathcal{F}$ is the Fourier transform the

$$H = \mathcal{F}^* \left(-\frac{d^2}{d\xi^2} + \xi^2\right) \mathcal{F}.$$

- $H \geq 3$???

One has to be careful with factorizations of operators. Indeed, let us consider

$$B^*B = \left(-\frac{d}{dx} + x - \frac{1}{x}\right)\left(\frac{d}{dx} + x - \frac{1}{x}\right)$$
$$= -\frac{d^2}{dx^2} + x^2 - 2 + \frac{1}{x^2} - 1 - \frac{1}{x^2} = -\frac{d^2}{dx^2} + x^2 - 3 \geq 0.$$

This implies

$$H = -\frac{d^2}{dx^2} + x^2 \geq 3.$$

Question: Where is a mistake?

Among the operators satisfying the property $H = \mathcal{F}^* H \mathcal{F}$ there are

$$H = -\frac{d^{2m}}{dx^{2m}} + x^{2m}, \qquad m \in \mathbb{N}.$$

In this lecture we shall consider the operator whose symbol equals

$$2\cosh\xi + 2\cosh x.$$

All such operators are positive and have discrete spectrum.

• Question: Is there a possibility to apply a version of the Darboux transform for such operators and find their spectrum.

- Coherent state transform.

Let us consider the map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ and defined by

$$\widetilde{\psi}(x,\xi) = (\Phi\,\psi)(x,\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi y} g(x-y)\psi(y)\,dy,$$

where

$$g(x) = (1/\pi)^{1/4} e^{-x^2/2}.$$

Note that $\int_{-\infty}^{\infty} g^2(x)dx = 1$ and

$$\begin{aligned}\Phi^*\Phi\,\psi(x) &= \int_{\mathbb{R}^3} e^{2\pi i\xi x} g(x-z) e^{-2\pi i\xi y} g(z-y)\psi(y)\,d\xi dy dz \\ &= \int_{\mathbb{R}^2} \delta(x-y) g(x-z) g(z-y)\,\psi(y)\,dydz \\ &= \psi(x)\int_{-\infty}^{\infty} g^2(x-z)\,dz = \psi(x).\end{aligned}$$

Theorem.

The map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is an isometry, such that $\Phi^*\Phi = I$ and $P = \Phi\Phi^*$ is an orthogonal projection in $L^2(\mathbb{R}^2)$.

• Action of the coherent state transform on the Harmonic oscillator.

Let us compute $\Phi^*\xi^2\Phi$.

$$
\begin{aligned}
(\Phi^*\xi^2\Phi\psi,\psi) &= \int_{\mathbb{R}^4} e^{2\pi i\xi x} g(x-z)\,\xi^2\,e^{-2\pi i\xi y} g(z-y)\psi(y)\overline{\psi(x)}\,d\xi dy dz dx \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{d}{dx}\left(e^{2\pi i\xi x}\right) g(x-z)\,\frac{d}{dy}\left(e^{-2\pi i\xi y}\right) g(z-y)\psi(y)\overline{\psi(x)}\,d\xi dy dz dx \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} e^{2\pi i\xi(x-y)}\frac{d}{dx}\Big(g(x-z)\overline{\psi(x)}\Big)\frac{d}{dy}\Big(g(z-y)\psi(y)\Big)\,d\xi dy dz dx \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\Big(\left(g'(x-z)\right)^2|\psi(x)|^2 + g^2(x-z)|\psi'(x)|^2\Big)\,dz dx \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}(g'(z))^2 dz\,\|\psi\|_2^2 + \frac{1}{4\pi^2}\|\psi'\|_2^2.
\end{aligned}
$$

Corollary.

$$
\|\psi'\|_2^2 = \int_{-\infty}^{\infty}(2\pi)^2\xi^2|\tilde{\psi}(z,\xi)|^2\,dz d\xi - \frac{1}{2}\|\psi\|_2^2.
$$

Similarly computing $\Phi^* z^2 \Phi$ we obtain

$$
\begin{aligned}
(\Phi^* z^2 \Phi\psi, \psi) &= \int_{\mathbb{R}^4} e^{2\pi i \xi x} g(x-z)\, z^2\, e^{-2\pi i \xi y} g(z-y)\psi(y)\overline{\psi(x)}\, d\xi dy dz dx \\
&= \int_{\mathbb{R}^2} z^2 g^2(x-z)|\psi(x)|^2\, dz dx = \int_{\mathbb{R}^2} (x-t)^2 g^2(t)|\psi(x)|^2\, dz dx \\
&= \int_{-\infty}^{\infty} x^2 |\psi(x)|^2\, dx + \int_{-\infty}^{\infty} t^2 g^2(t)\, dt \int_{-\infty}^{\infty} |\psi(x)|^2\, dx.
\end{aligned}
$$

Corollary

$$
\|x\,\psi\|_2^2 = \int_{-\infty}^{\infty} z^2 |\widetilde{\psi}(z,\xi)|^2\, dz d\xi - \frac{1}{2}\|\psi\|_2^2.
$$

Proposition. There is the following representation of the quadratic form of the Harmonic oscillator H

$$
(H\psi, \psi) = \int_{-\infty}^{\infty} \left((2\pi)^2 \xi^2 + z^2\right) |\widetilde{\psi}(z,\xi)|^2\, dz d\xi - \|\psi\|_2^2.
$$

• Further properties of the coherent state transform.

Let us introduce the convolution

$$\varphi * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)\,dy.$$

and let $\mathcal{F}$ be the Fourier transform

$$\widehat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int_{-\infty}^{\infty} e^{2\pi i\xi x}\psi(x)\,dx.$$

Lemma.

$$(\Phi\psi)(x,\xi) = \widetilde{\psi}(x,\xi) = e^{-2\pi i x\xi} \int_{-\infty}^{\infty} \widehat{\psi}(\eta)\, e^{2\pi i\eta x}\overline{\widehat{g}(\eta-\xi)}\,d\eta \qquad (*)$$

and

$$\int_{-\infty}^{\infty} |\widetilde{\psi}(x,\xi)|^2 dx = \int_{-\infty}^{\infty} |\widehat{\psi}(\xi-\eta)|^2|\widehat{g}(\eta)|^2\,d\eta = |\widehat{\psi}|^2 * |\widehat{g}|^2(\xi). \quad (**)$$

$$\int_{\mathbb{R}} |\widetilde{\psi}(x,\xi)|^2\,d\xi = \int_{-\infty}^{\infty} |\psi|(x-y)|^2|g(y)|^2\,dy = (|\psi|^2 * |g|^2)(x) \qquad (***)$$

Proof. Let us first show $(*)$

$$
\begin{aligned}
e^{-2\pi i x\xi} \int_{-\infty}^{\infty} \widehat{\psi}(\eta)\, e^{2\pi i\eta x}\overline{\widehat{g}(\eta-\xi)}\, d\eta& \\
&= e^{-2\pi i x\xi} \int_{\mathbb{R}^3} e^{-2\pi i\eta z}\psi(z)e^{2\pi i\eta x}e^{2\pi i(\eta-\xi)t}g(t)\, dt dz d\eta \\
&= e^{-2\pi i x\xi} \int_{\mathbb{R}^2} \delta(t+x-z)\psi(z)e^{-2\pi i\xi t}g(t)\, dt dz \qquad [t=z-x] \\
&= \int_{-\infty}^{\infty} e^{-2\pi i\xi z}\psi(z)g(x-z)\, dz = \widetilde{\psi}(x,\xi).
\end{aligned}
$$

In order to obtain $(**)$ we write by using $(*)$

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\widetilde{\psi}(x,\xi)|^2 dx = \int_{\mathbb{R}^3} \overline{\widehat{\psi}(\rho)}\, e^{-2\pi i\rho x}\widehat{g}(\rho-\xi)\ \widehat{\psi}(\eta)\, e^{2\pi i\eta x}\overline{\widehat{g}(\eta-\xi)}\, dx d\rho d\eta& \\
= \int_{-\infty}^{\infty} |\widehat{\psi}(\eta)|^2 |\widehat{g}(\eta-\xi)|^2\, d\eta.&
\end{aligned}
$$

Exercise. Prove $(***)$.

• Weyl operators.

We now consider a class of functional discrete operators that have some analogy with Harmonic oscillators, but whose spectrum is more complicated.

Let

$$U\psi(x) = \psi(x+i) \qquad \text{and} \qquad V\psi(x) = e^{2\pi x}\psi(x).$$

Then

$$UV\psi(x) = e^{x+i}\psi(x+i) = e^{i}VU\psi(x).$$

The respective domains of these operators are

$$D(U) = \left\{\psi \in L^2(\mathbb{R}) : \, e^{-2\pi\xi}\,\widehat{\psi}(\xi) \in L^2(\mathbb{R})\right\}$$

and

$$D(V) = \left\{\psi \in L^2(\mathbb{R}) : \, e^{2\pi x}\,\psi(x) \in L^2(\mathbb{R})\right\}.$$

Equivalently, $D(U)$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip

$$\{z = x + iy \in \mathbb{C} :\ 0 < y < 1\}$$

such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leq y < 1$ and there is a limit

$$\psi(x + i - i0) = \lim_{\varepsilon \to 0^+} \psi(x + i - i\varepsilon)$$

in the sense of convergence in $L^2(\mathbb{R})$, which we will denote by $\psi(x + i)$.

Question: Prove it.

The domains of U^{-1} and V^{-1}can be characterised similarly and obviously

$$U^{-1}\psi(x) = \psi(x - i) \qquad \text{and} \qquad V^{-1}\psi(x) = e^{-2\pi x}\psi(x).$$

Our main object of study is the operator H

$$H = U + U^{-1} + V + V^{-1}$$

whose symbol is $2\cosh\xi + 2\cosh x$.

Remark.

It was discovered by M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, that the functional-difference operators built from the Weyl operators U and V, appear in the study of local mirror symmetry as a quantisation of an algebraic curve, the mirror to a toric Calabi-Yau threefold. The spectral properties of these operators were considered in A. Grassi, Y. Hatsuda, and M. Marino.

Remark. The operator

$$\mathcal{H}\psi(x) = (U + U^{-1} + V)\psi(x) = \psi(x+i) + \psi(x-i) + e^{2\pi x}\psi(x)$$

first appeared in the study of the quantum Liouville model on the lattice and plays an important role in the representation theory of the non-compact quantum group $SL_q(2;\mathbb{R})$. In the momentum representation it becomes the Dehn twist operator in quantum Teichmuller theory.

In particular, R. Kashaev obtained the eigenfunction expansion theorem for this operator in the momentum representation. It was stated as formal completeness and orthogonality relations in the sense of distributions. The spectral analysis of the functional-difference operator $\mathcal{H}$ was done in the recent paper of L. D. Faddeev and L. A. Takhtajan. The operator $\mathcal{H}$ was shown to be self-adjoint with a simple absolutely continuous spectrum $[2, \infty)$, and the authors proved eigenfunction expansion theorem for $\mathcal{H}$, by generalizing the classical Kontorovich-Lebedev transform.

• Action of the coherent state transform on functional discrete operators.

We aim to find representations of $(U\psi, \psi)$ and $(V\psi, \psi)$ in terms of coherent states.
It follows from $(**)$ that

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\, d\xi\, dx = \iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widehat{\psi}(\xi-\eta)|^2|\widehat{g}(\eta)|^2\, d\xi\, d\eta,$$

and using

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\, d\xi\, dy \\
&\qquad = \iint_{\mathbb{R}^2} 2\cosh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta \\
&\qquad\qquad + \iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta\,.
\end{aligned}$$

The first integral on the right-hand side can be computed to be

$$((U+U^{-1})\psi,\psi)((V+V^{-1})\widehat{g},\widehat{g}).$$

$$\iint_{\mathbb{R}^2} 2\cosh\big(2\pi(\xi-\eta)\big)\,|\widehat{\psi}(\xi-\eta)|^2 \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta \qquad (*)$$

$$+ \iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)\,|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta\,.$$

Indeed, note that

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{2\pi i x\xi}\, 2\cosh(2\pi\xi)\,\widehat{\psi}(\xi)\,d\xi &= \iint_{\mathbb{R}^2} e^{2\pi i x\xi}\left(e^{-2\pi\xi}+e^{2\pi\xi}\right)\widehat{\psi}(\xi)\,d\xi \\ &= \iint_{\mathbb{R}^2}\left(e^{2\pi i(x+i)\xi}+e^{2\pi i(x-i)\xi}\right)\widehat{\psi}(\xi)\,d\xi \\ &= \psi(x+i)-\psi(x-i) = (U+U^{-1})\psi(x).\end{aligned}$$

Therefore the first integral $(*)$ equals

$$((U+U^{-1})\psi,\psi)\int_{\mathbb{R}} \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\,d\eta = ((U+U^{-1})\psi,\psi)\,((V+V^{-1})\widehat{g},\widehat{g})/2.$$

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2 \, d\xi \, dy$$

$$= \iint_{\mathbb{R}^2} 2\cosh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \cosh(2\pi\eta)|\widehat{g}(\eta)|^2 \, d\xi \, d\eta$$

$$+ \iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2 \, d\xi \, d\eta \, . \qquad (**)$$

Since $g(x) = g(-x)$, it holds that $\widehat{g}(\xi) = \widehat{g}(-\xi)$ and consequently the integral (**)

$$\iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2 \, d\xi \, d\eta$$

vanishes.

Thus for $\psi \in D(U)$ we obtain the representation

$$((U+U^{-1})\psi,\psi) = d_1 \iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2 \, d\xi \, dx$$

where

$$d_1 = \frac{2}{((V+V^{-1})\widehat{g},\widehat{g})} = e^{-1/4} < 1.$$

Similarly, we can use $(***)$ to compute that

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx = \iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\psi(x-z)|^2|g(z)|^2\,dx\,dz,$$

which with the help of the same trigonometric identity as above can be simplified to

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx = ((V+V^{-1})\psi,\psi)\,((V+V^{-1})g,g)/2.$$

Thus for $\psi \in D(V)$ we have the representation

$$((V+V^{-1})\psi,\psi) = d_2 \iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx,$$

where

$$d_2 = \frac{2}{((V+V^{-1})g,g)} = e^{-\pi^2} < 1.$$

• Summary: Coherent state representation for H.

Summarising, we obtain a remarkable identity

$$(H\psi, \psi) = ((U + U^{-1})\psi, \psi) + ((V + V^{-1})\psi, \psi)$$
$$= \iint_{\mathbb{R}^2} 2(d_1 \cosh(2\pi\xi) + d_2 \cosh(2\pi x))|\widetilde{\psi}(x, \xi)|^2 \, d\xi \, dx.$$

• Deriving an Upper Bound.

Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of H and let $\{\psi_j\}_{j=1}^{\infty}$ be the corresponding orthonormal eigenfunctions which form a complete set. We first observe that the coherent state representation of H yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}\left(\lambda-(H\psi_j,\psi_j)\right)_+$$

$$= \sum_{j\geq 1}\left(\lambda - \iint_{\mathbb{R}^2} 2\big(d_1\cosh(2\pi\xi)+d_2\cosh(2\pi x)\big)|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dy\right)_+ .$$

Note

$$\iint_{\mathbb{R}^2}|\widetilde{\psi}_j(x,\xi)|^2\,dx d\xi = \|\psi_j\|_2^2 = 1.$$

Therefore

$$\begin{aligned}
&\sum_{j\geq 1}(\lambda-\lambda_j)_+ \\
&\quad = \sum_{j\geq 1}\left(\iint_{\mathbb{R}^2}(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x))\,|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\right)_+ \\
&\quad\quad \leq \iint_{\mathbb{R}^2}\big(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x)\big)_+\sum_{j\geq 1}|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\,.
\end{aligned}$$

Denote $e_{x,\xi}(y)=e^{2\pi i\xi y}g(x-y)$. Since the eigenfunctions ψ_j form an orthonormal basis in $L^2(\mathbb{R})$

$$\sum_{j=1}^{\infty}|\widetilde{\psi}_j(x,\xi)|^2=\sum_{j=1}^{\infty}|(e_{x,\xi},\psi_j)|^2=\|e_{x\xi}\|^2=1\quad\text{for all}\quad x,\xi\in\mathbb{R},$$

we arrive at the upper bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+\leq\iint_{\mathbb{R}^2}\big(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x)\big)_+\,d\xi\,dx\,.$$

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \iint_{\mathbb{R}^2} \big(\lambda - 2d_1\cosh(2\pi\xi) - 2d_2\cosh(2\pi x)\big)_+ \, d\xi\, dx\,.$$

To investigate the behaviour of the integral on the right-hand side as $\lambda \to \infty$, we first note that

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &\leq 4\int_0^\infty \int_0^\infty \big(\lambda - 2d_1\cosh(2\pi\xi) - 2d_2\cosh(2\pi x)\big)_+ \, d\xi\, dx \\ &\leq 4\int_0^\infty \int_0^\infty \big(\lambda - d_1 e^{2\pi\xi} - d_2\, e^{2\pi x}\big)_+ \, d\xi\, dx\,,\end{aligned}$$

where we used that $2\cosh x > e^x$ for $x > 0$.

Changing the variables $u_1 = d_1 e^{2\pi\xi}, u_2 = d_2 e^{2\pi x}$ we arrive at

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &\leq \frac{1}{\pi^2}\int_{d_1}^\infty \int_{d_2}^\infty \frac{(\lambda - u_1 - u_2)_+}{u_1 u_2}\, du_2\, du_1 \\ &= \frac{1}{\pi^2}\int_{d_1}^{\lambda-d_2} \int_{d_2}^{\lambda-u_1} \frac{\lambda - u_1 - u_2}{u_1 u_2}\, du_2\, du_1\,.\end{aligned}$$

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{1}{\pi^2}\int_{d_1}^{\lambda-d_2}\int_{d_2}^{\lambda-u_1}\frac{\lambda-u_1-u_2}{u_1u_2}\,du_2\,du_1\,.$$

Here $\lambda \geq d_1+d_2$ since $\lambda\geq 2$ and $d_1, d_2 \leq 1/2$. Now we immediately obtain

$$\begin{aligned}\int_{d_1}^{\lambda-d_2}\int_{d_2}^{\lambda-u_1}\frac{\lambda-u_1-u_2}{u_1u_2}\,du_2\,du_1 &= \lambda\int_{d_1/\lambda}^{1-d_2/\lambda}\int_{d_2/\lambda}^{1-v_1}\frac{1-v_1-v_2}{v_1v_2}\,dv_2\,dv_1\\ &= \lambda\log^2\lambda + O(\lambda\log\lambda)\end{aligned}$$

as $\lambda\to\infty$, so that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{\lambda\log^2\lambda}{\pi^2} + O(\lambda\log\lambda).$$

• Deriving a Lower Bound.

To obtain a lower bound, we use a different argument. Since $\|\psi_j\|_2 = \|\widetilde{\psi}_j\|_2 = 1$ we start from the identity

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}(\lambda-\lambda_j)_+ \iint_{\mathbb{R}^2} |\widetilde{\psi}_j(x,\xi)|^2\, dx d\xi,$$

and observe that, if as before $e_{x,\xi}(y) = e^{2\pi i\, y\, \xi} g(x-y)$, we have

$$\widetilde{\psi}_j(x,\xi) = \int_{\mathbb{R}} \psi_j(y)\overline{e_{x,\xi}(y)}\, dy = (\psi_j, e_{x,\xi}).$$

This implies

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &= \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ (\psi_j, e_{x,\xi})\overline{(\psi_j, e_{x,\xi})}\, d\xi dx \\ &= \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ \big((e_{x,\xi},\psi_j)\psi_j, e_{x,\xi}\big)\, d\xi dx\,.\end{aligned}$$

Denoting by dE_μ the projection-valued spectral measure for H on $[2,\infty)$, we conclude that

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &= \iint_{\mathbb{R}^2}\sum_{j\geq 1}(\lambda-\lambda_j)_+\big((e_{x,\xi},\psi_j)\psi_j, e_{x,\xi}\big)\, d\xi dx \\ &= \iint_{\mathbb{R}^2}\int_2^\infty(\lambda-\mu)_+(dE_\mu e_{x,\xi}, e_{x,\xi})\, d\xi dx\,.\end{aligned}$$

Since by the spectral theorem

$$\int_2^\infty (dE_\mu e_{x,\xi}, e_{x,\xi}) = (e_{x,\xi}, e_{x,\xi}) = \|g\|_2^2 = 1,$$

we can apply Jensen's inequality with the convex function $x \mapsto (\lambda - x)_+$ and obtain the lower bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda - \int_2^\infty \mu\,(dE_\mu\, e_{x,\xi}, e_{x,\xi})\right)_+ d\xi dx\,.$$

Computing

$$\int_2^\infty \mu \left(dE_\mu \, e_{x,\xi}, e_{x,\xi}\right) d\xi dx.$$

It follows from the spectral theorem that

$$\begin{aligned}\int_2^\infty \mu \left(dE_\mu \, e_{x,\xi}, e_{x,\xi}\right) &= \left(H e_{x,\xi}, e_{x,\xi}\right) \\ &= ((U+U^{-1})e_{x,\xi}, e_{x,\xi}) + ((V+V^{-1})e_{x,\xi}, e_{x,\xi})\,.\end{aligned}$$

The two terms on the right-hand side can be computed explicitly.
We first note that

$$g(x-y\pm i) = (1/\pi)^{1/4}\, e^{(x-y\pm i)^2} = e^{1/2} g(x-y) e^{\mp(x-y)i},$$

whence

$$\begin{aligned}((U+U^{-1})e_{x,\xi}, e_{x,\xi}) &= \int_{-\infty}^\infty \big(e^{-2\pi\xi} g(x-y+i) + e^{2\pi\xi} g(x-y-i)\big) g(x-y)\, dy \\ &= e^{1/2}\left(e^{-2\pi\xi}\int_{-\infty}^\infty g(z)^2 e^{-iz}\, dz + e^{2\pi\xi}\int_{-\infty}^\infty g(z)^2 e^{iz}\, dz\right) \\ &= \frac{1}{d_1} 2\cosh(2\pi\xi)\,.\end{aligned}$$

For the second term, $((V+V^{-1})e_{x,\xi},e_{x,\xi})$, we get

$$
\begin{aligned}
((V+V^{-1})e_{x,\xi},e_{x,\xi}) &= \int_{-\infty}^{\infty} 2\cosh(2\pi y)g(x-y)^2\,dy \\
&= \int_{-\infty}^{\infty} 2\cosh\big(2\pi(x-y)\big)\cosh(2\pi y)g(x-y)^2\,dy \\
&\quad + \int_{-\infty}^{\infty} 2\sinh\big(2\pi(x-y)\big)\sinh(2\pi b y)g(x-y)^2\,dy = \frac{1}{d_2}2\cosh(2\pi x)\,.
\end{aligned}
$$

Therefore we finally arrive at

$$
\begin{aligned}
\sum_{j\ge 1}(\lambda-\lambda_j)_+ &\ge \iint_{\mathbb{R}^2}\left(\lambda-\frac{2}{d_1}\cosh(2\pi\xi)-\frac{2}{d_2}\cosh(2\pi x)\right)_+ d\xi dx \\
&= 4\int_0^{\infty}\int_0^{\infty}\left(\lambda-\frac{2}{d_1}\cosh(2\pi\xi)-\frac{2}{d_2}\cosh(2\pi x)\right)_+ d\xi dx\,.
\end{aligned}
$$

Note that $2\cosh x \le 2e^x$ for $x \ge 0$ and thus

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ \ge 4\int_0^\infty\int_0^\infty \left(\lambda - \frac{2}{d_1}e^{2\pi\xi} - \frac{2}{d_2}e^{2\pi x}\right)_+ d\xi dx\,.$$

The integral on the right-hand side is computed in the same way as previously. The only difference is that the numbers d_1, d_2 have been replaced by $2/d_1, 2/d_2$. These coefficients have no influence on the leading term for large λ as long as $\lambda \ge 2/d_1 + 2/d_2$, and we conclude

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ \ge \frac{1}{\pi^2}\lambda\log^2\lambda + O(\lambda\log\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Thus we have the following result

Theorem. For the Riesz mean of the eigenvalues of the operator H we have

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ = \frac{1}{\pi^2}\lambda\log^2\lambda + O(\lambda\log\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Corollary. For the number $N(\lambda) = \#\{j\in\mathbb{N} :\, \lambda_j < \lambda\}$ of eigenvalues of the operator H below λ we have

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\log^2\lambda} = \pi^{-2}.$$

- Open Problems.

1. The symbol of the operator H equals

$$2\cosh\xi + 2\cosh x = 2\sum_{n=0}^{\infty}\frac{1}{(2n)!}(\xi^{2n} + x^{2n}).$$

Therefore using that the first eigenvalue of the Harmonic oscillator equals 1 we have $\lambda_1(H) \geq 3$. Find λ_1.

2. Estimate from below the first eigenvalue of the operator

$$H_n = (-1)^n D^{2n} + x^{2n}.$$

Thank you